Summary of the Ph.D. thesis

Integrality, complexity and colourings in polyhedral combinatorics

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1 Complexity of Hilbert bases, TDL-ness and g-polymatroids

In Chapter 2 of the thesis we address the complexity of several properties of polyhedra, linear systems or sets of vectors, and show some related results as by-products.

1.1 Testing Hilbert bases is hard

A finite set of integer vectors is called a *Hilbert basis* if every integer vector in their cone can be written as a nonnegative integral combination of them. In 1990, Papadimitriou and Yannakakis [15] raised the questions what the complexity of the recognition problems of TDI systems and Hilbert bases is. Both were open for a long time. Recently Ding, Feng and Zang [11] proved that the problem "Is $Ax \geq 1,\ x \geq 0$ TDI?" is co-NP-complete, even if A is the incidence matrix of a graph. We answer the other question, strengthening this result.

Theorem 1.1 ([7]). *The problem of deciding whether or not a set of integer vectors forms a Hilbert basis is co-NP-complete even if the set consists of 0–1 vectors having at most three ones.*

1.2 Total dual laminarity and generalized polymatroids

Consider a packing-covering polyhedron

$$Q(p,b) := \{x \in \mathbb{R}^n : p(S) \leq x(S) \leq b(S)\ \forall S \subseteq [n]\} \tag{1}$$

for set functions $p : 2^{[n]} \to \mathbb{R} \cup \{-\infty\}$ and $b : 2^{[n]} \to \mathbb{R} \cup \{+\infty\}$. Note that infinities mean absent constraints. The pair (p,b) is *paramodular* if p is supermodular, b is submodular, $p(\emptyset) = b(\emptyset) = 0$, and the "cross-inequality" $b(S) - p(T) \geq b(S \setminus T) - p(T \setminus S)$ holds for all $S, T \subseteq [n]$. A *generalized polymatroid* is any polyhedron $Q(p,b)$ where (p,b) is paramodular; and also the empty set is considered to be a g-polymatroid.

The dual variables correspond to sets with finite p- or b-value. The *support* of a dual solution is the set system consisting of all sets for which at least one dual variable is nonzero. A dual solution is *laminar* if its support is a laminar set system (that is, each two members are disjoint or one contains the other).

Definition 1.2. The pair (p,b) is *totally dual laminar* (TDL) if for every primal objective with finite optimal value, some optimal dual solution is laminar.

Theorem 1.3 ([1]). *Deciding whether a given system is TDL is NP-hard.*

Theorem 1.4 ([1]). *If P is a polyhedron whose intersection with each integral g-polymatroid is integral, then P is an integral g-polymatroid.*

1.3 Recognizing Generalized Polymatroids

Theorem 1.5 ([1]). *There is a polynomial-time algorithm that, on input (A,b), determines whether the polyhedron $\{x :\ Ax \leq b\}$ is a g-polymatroid.*

Corollary 1.6. *Let $P = \{x :\ Ax \leq b\} \cap \{x :\ x([n]) = c\}$. Then we can test in polynomial time whether P is a base polyhedron.*

The full-dimensional case of Theorem 1.5 can be accomplished using the following new characterization. It is enough to consider systems of the form (1). Let $\mathcal{B}$ be the family of all S where $x(S) \le b(S)$ is part of the input (that is, $b(S) \neq +\infty$). Similarly let $\mathcal{P}$ be the family of all S where $x(S) \ge p(S)$ is part of the input.

Theorem 1.7 ([1])**.** *Suppose that for (p, b), the polyhedron $Q(p, b)$ is full-dimensional and let $i(S) := \min_{x \in P} x(S)$ and $a(S) := \max_{x \in P} x(S)$. Then $Q(p, b)$ is a g-polymatroid if and only if*

(i) for every $S, T \in \mathcal{B}$, $a(S \cup T) + a(S \cap T) \le b(S) + b(T)$ holds,

(ii) for every $S, T \in \mathcal{P}$, $i(S \cup T) + i(S \cap T) \ge p(S) + p(T)$ holds, and

(iii) for every $S \in \mathcal{B}$ and $T \in \mathcal{P}$, $a(S \setminus T) - i(T \setminus S) \le b(S) - p(T)$ holds.

Corollary 1.8. *If $Q(p, b)$ is a full-dimensional g-polymatroid, then (p, b) is TDL.*

We can also decide whether a given linear system of the form (1) describes an integer g-polymatroid.

Theorem 1.9 ([1])**.** *Suppose that $Q(p, b)$ is a g-polymatroid, and that it is minimally described. Then $Q(p, b)$ is an integer g-polymatroid if and only if p and b are integral and for every set on which the sum is constant in $Q(p, b)$, the sum is integer.*

2 Polyhedral Sperner's Lemma and applications

We begin Chapter 3 of the thesis with stating polyhedral variants of the multidimensional Sperner Lemma. For a colouring of the vertices of a polytope P, a facet of P is *multicoloured* if it contains vertices of every colour. For a colouring of the facets of P, a vertex of P is *multicoloured* if it lies on facets of every colour. A *simple vertex* of an n-dimensional polyhedron is a vertex that lies on exactly n facets.

Theorem 2.1. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P with n colours such that F_0 is multicoloured. Then there is another multicoloured facet.*

Theorem 2.2. *Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If the facets of the polyhedron are coloured with n colours such that facets containing the i-th extreme direction do not get colour i, then there is a multicoloured vertex.*

2.1 Using the polyhedral Sperner Lemma

By applying Theorem 2.2 we get short proofs of several purely combinatorial and some game-theoretical results. Most of these results are known, but some of them and the method are new and joint work with Király [2, 3].

Kernels in directed graphs

Definition 2.3. In a directed graph $D = (V, A)$, a stable set $S \subseteq V$ is said to be a *kernel* if from every node of $V \setminus S$ there is an arc to S. Let $G = (V, E)$ be an undirected graph. A *superorientation* of G is a directed graph $\vec{G}$ obtained by replacing each edge uv of G by an arc uv or an arc vu or both. A *one-way cycle* in a superorientation is a directed cycle consisting of arcs that are not present reversed in the digraph. We define a *source node* of an induced subdigraph $\vec{G}[U]$ as a node in U from which there are arcs to all of its neighbours in $G[U]$. A superorientation is *clique-acyclic* if no clique contains a one-way cycle (equivalently, if every clique contains a source node). A graph G is *kernel solvable* if every clique-acyclic superorientation of G has a kernel.

Berge and Duchet [9] conjectured that a graph is kernel solvable if and only if it is perfect. The kernel-solvability of perfect graphs was proved by Boros and Gurvich [10], who used elaborate game-theoretic machinery. We give a short proof that relies on Theorem 2.2.

Theorem 2.4 (Boros, Gurvich [10])**.** *Every perfect graph is kernel solvable.*

We extend Theorem 2.4 to arbitrary undirected graphs, provided some conditions hold that depend on the facets of $\mathrm{STAB}(G)$. Let $\vec{G}$ be a superorientation of a graph $G = (V, E)$, and let U be a set of vertices. We say that $\vec{G}$ is *one-way-cycle-free* (or *owc-free*) *in* U if there is no one-way cycle in $\vec{G}[U]$.

Theorem 2.5 ([2])**.** *If* $\mathrm{STAB}(G) = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ *for nonnegative* A *and* b*, and* $\vec{G}$ *is a superorientation of* G *which is owc-free in* $\mathrm{supp}(a)$ *for every row* a *of* A*, then there is a kernel in* $\vec{G}$*.*

A graph is called *h-perfect* if its stable set polytope is described by the following set of inequalities:

$$\begin{aligned} x_v &\geq 0 && \text{for every } v \in V, \\ x(C) &\leq 1 && \text{for every maximal clique } C, \\ x(Z) &\leq \frac{|Z|-1}{2} && \text{for every odd hole } Z. \end{aligned}$$

To apply Theorem 2.5 to h-perfect graphs, let us call a superorientation of a graph *odd-hole-acyclic* if no oriented odd hole is a one-way cycle.

Theorem 2.6 ([2])**.** *If* $\vec{G}$ *is a superorientation of an h-perfect graph and is clique-acyclic and odd-hole-acyclic, then it has a kernel.*

A digraph is *kernel-perfect* if all of its induced subdigraphs have kernels.

Corollary 2.7. *It is in co-NP to decide whether a given superorientation of an h-perfect graph is kernel-perfect.*

Scarf's Lemma

Scarf [16] proved that a balanced n-person game with non-transferable utilities (NTU) always has a non-empty core. His proof relies on a lemma in which we consider a bounded polyhedron $P = \{x \in \mathbb{R}^n : Ax \le b,\ x \ge 0\}$ where A is an $m \times n$ nonnegative matrix (with non-zero columns) and $b \in \mathbb{R}^m$ is a positive vector. In addition, for every row $i \in [m]$ of A, a total order $<_i$ of the columns (or a subset of them) is given. We denote the domain of $<_i$ by $\mathrm{Dom}(<_i)$. If $j \in \mathrm{Dom}(<_i)$ and $K \subseteq \mathrm{Dom}(<_i)$, we use the notation $j \le_i K$ as an abbreviation for "$j \le_i k$ for every $k \in K$".

Definition 2.8. A vertex x^* of P *dominates* column j if there is a row i where $a_i x^* = b_i$ and $j \le_i \mathrm{supp}(x^*) \cap \mathrm{Dom}(<_i)$ (this implies that $j \in \mathrm{Dom}(<_i)$). A vertex x^* of P is *maximal* if by increasing any coordinate of x^* we leave P (or formally, $(\{x^*\} + \mathbb{R}^n_+) \cap P = \{x^*\}$).

Theorem 2.9 (Scarf's Lemma [16])**.** *Let P be as above and let $<_i$ be a total order on supp(a_i) $(i \in [m])$, where a_i is the i-th row of A. Then P has a maximal vertex that dominates every column.*

We show that essentially, Scarf's Lemma for a polyhedron P corresponds to the polyhedral Sperner's Lemma 2.2 for the polyhedron $P - \mathbb{R}^n_+$. This gives a new proof of Scarf's Lemma.

Fractional core of NTU games

In a *finitely generated non-transferable utility (NTU) game* there are m *players*, and a finite multiset of *basic coalitions* $S_j \subseteq [m]$ $(j \in [n])$. Each player i has a total order $<_i$ of the basic coalitions that he participates in; $S_j <_i S_k$ means that the player i prefers coalition S_k to coalition S_j.

A set $\mathcal{S}$ of basic coalitions is said to be in the *core* of the game if they are disjoint and for each basic coalition S' not in $\mathcal{S}$ there is a player $i \in S'$ and a basic coalition $S \in \mathcal{S}$ such that $S' <_i S$.

A vector $x \in \mathbb{R}^n_+$ is in the *fractional core* if for each player i, $\sum_{j: i \in S_j} x(j) \le 1$, and for each $j \in [n]$ there is a player i in S_j such that $\sum_{k: i \in S_k} x(k) = 1$ and $S_j \leqslant_i S_k$ whenever $i \in S_k$ and $x(k) > 0$. A vector $x \in \mathbb{R}^n_+$ is *admissible* if $\sum_{j: i \in S_j} x(j) \le 1$ for every player i. Using Theorem 2.2 we give a short proof of the following version of Scarf's Theorem.

Theorem 2.10 (Scarf [16])**.** *The fractional core of a finitely generated NTU-game is always non-empty. If the polyhedron of admissible vectors is integer, then the core is also non-empty.*

A matroidal generalization of kernels

Fleiner [12] defined a notion of matroid-kernels. An *ordered matroid* $\mathcal{M} = \{S, \mathcal{I}, <\}$ is a matroid on ground set S with independent sets $\mathcal{I}$ together with a linear order $<$ of its ground set. In an ordered matroid, a set $X \subseteq S$ is said to *dominate* an element e if either $e \in X$ or there is an independent set $Y \subseteq X$ for which $Y \cup \{e\} \notin \mathcal{I}$ (that is, Y spans e) and $e < y$ for each $y \in Y$.

Matroid-kernels concern two ordered matroids on the same ground set – let $\mathcal{M}_1 = \{S, \mathcal{I}_1, <_1\}$ and $\mathcal{M}_2 = \{S, \mathcal{I}_2, <_2\}$ be ordered matroids. A set $K \subset S$ is called an $\mathcal{M}_1\mathcal{M}_2$-*kernel* if it is a common independent set of the two matroids (that is, $K \in \mathcal{I}_1 \cap \mathcal{I}_2$) and every element $e \in S$ is dominated by K in (at least) one of the two matroids.

Theorem 2.11 (Fleiner [12]). *For every two ordered matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, there exists an $\mathcal{M}_1\mathcal{M}_2$-kernel.*

We give a short proof using Theorem 2.2 on the downhull of the matroid intersection polytope.

Orientation of clutters

Ideal clutters are Sperner systems of which the covering polyhedron is integral. Using our method we get the following new result for clutters.

Theorem 2.12. *Let $\mathcal{A}$ be an ideal clutter on ground set $[n]$ and let $\mathcal{B}$ be its blocker. Then there are no functions $p : \mathcal{A} \to S$ and $q : \mathcal{B} \to S$ such that $p(A) \in A\ \forall A \in \mathcal{A}$, $q(B) \in B\ \forall B \in \mathcal{B}$ and if $p(A) = q(B)$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $|A \cap B| > 1$.*

Stable flows

In an instance of the stable flow problem we have a network on digraph $D = (V, A)$ with $s, t \in V$ and capacities $c \in \mathbb{R}_+^A$, and additionally linear orders $\leq_v$ for each node v on the arc set incident to v. The network along with the set of these preference orders is called a *network with preferences*.

Let f be a flow of network (D, s, t, c). A *rooted cycle* is a directed cycle in which one node is designated as the root. It can be regarded as a path which ends at its starting node. A path or rooted cycle $P = (v_1, a_1, v_2, a_2, \dots, a_{k-1}, v_k)$ is said to *block* f if the following hold:

(i) $v_i \neq s, t$ if $i \in \{2, 3, \dots, k-1\}$,
(ii) each arc a_i is unsaturated in f,
(iii) $v_1 = s$ or $v_1 = t$ or there is an arc $a' = v_1 u$ for which $f(a') > 0$ and $a' <_{v_1} a_1$,
(iv) $v_k = s$ or $v_k = t$ or there is an arc $a'' = wv_k$ for which $f(a'') > 0$ and $a'' <_{v_1} a_{k-1}$.

A flow is called *stable* if there is no path or cycle blocking it. We give a new proof of Fleiner's following result, using Theorem 2.2 on a polyhedron of exponential dimension.

Theorem 2.13 (Fleiner [13]). *In every network with preferences there exists a stable flow. If the capacity function is integral, then there is an integral stable flow.*

2.2 Attempts at converse statements

The Strong Perfect Graph Theorem implies the converse of the theorem of Boros and Gurvich 2.4. So the question arises, whether the converse of the other applications of Theorem 2.2 is also true, or even the converse of the theorem itself. We show that the converse of Theorem 2.6 does not hold.

Proposition 2.14 ([3]). *There exists a graph which is not h-perfect, but every clique- and odd-hole-acyclic superorientation of it has a kernel.*

It would be tempting to formulate a more general conjecture, which is a kind of converse to the polyhedral Sperner's Lemma. We give a counterexapmle.

Proposition 2.15 ([3]). *There exists a 4-dimensional polytope P, and two distinct vertices x^1 and x^2 of P, where x^1 is simple, such that the facets of P can not be coloured by 4 colours so that x^1 and x^2 are precisely the vertices that are incident to facets of all colours.*

Using this, a counterexample can be given to a converse of Scarf's Lemma.

Proposition 2.16 ([3]). *There is a nonnegative $m \times n$ matrix A, a positive vector $b \in \mathbb{R}^m$ for which the polyhedron $P = \{x : \ Ax \leq b, \ x \geq 0\}$ is bounded, and a maximal vertex x^* of P, such that no matter how we give a total order on $supp(a_i)$ for each row a_i of A, there is a maximal vertex different from x^* which dominates every column.*

Theorem 2.17. *The converse of Theorem 2.12 holds if $\mathcal{A}$ has an mni minor whose core is cyclic.*

2.3 PPAD-completeness

The following are the computational versions of Theorem 2.1 and Theorem 2.2.

POLYTOPAL SPERNER:

Input: vectors $v^i \in \mathbb{Q}^n$ $(i = 1, \dots, m)$ whose convex hull is a full-dimensional polytope P; a colouring of the vertices by n colours; a multicoloured simplex facet F_0 of P.

Output: n affine independent vectors $v^{i_1}, \dots, v^{i_n}$ with different colours which lie on a facet of P different from F_0.

EXTREME DIRECTION SPERNER:

Input: matrix $A \in \mathbb{Q}^{m \times n}$ and vector $b \in \mathbb{Q}^m$ such that $P = \{x : Ax \leq b\}$ is a pointed polyhedron whose characteristic cone is generated by n linearly independent vectors; a colouring of the facets by n colours such that facets containing the i-th extreme direction do not get colour i.

Output: a multicoloured vertex of P.

Theorem 2.18 ([5]). *Both* EXTREME DIRECTION SPERNER *and* POLYTOPAL SPERNER *are PPAD-complete.*

3 Ideal set functions

In Chapter 4 we extend of the notions of the blocking relation and idealness of clutters to set functions and prove that several properties are maintained. We also show new types of minimally non-ideal structures. This is joint work with Tamás Király [4].

3.1 Gradual set functions

Let V be a finite ground set, and let $f : 2^V \to \mathbb{Z}$ be an integer-valued set function that satisfies $f(X) \leq f(X + v) \leq f(X) + 1$ for every $X \subsetneq V$ and $v \in V \setminus X$. We call such a set function *gradual*. Let the *blocker* $b(f) : 2^V \to \mathbb{Z}$ of f be the (gradual) set function defined by $b(f)(X) := -f(V \setminus X)$ for any set $X \subseteq V$. Obviously, $b(b(f)) = f$.

We define the following two minor operations on gradual functions, resulting in gradual functions on ground set $V - v$ for a node $v \in V$:

deletion: $f\backslash v(X) := f(X)$ for every $X \subseteq V - v$,
contraction: $f/v(X) := f(X+v)$ for every $X \subseteq V - v$.

A function f' is a *minor* of f if it can be obtained from f using deletions and contractions.

Proposition 3.1. *If f is gradual, then its minors are also gradual.*

Proposition 3.2. *For any gradual function f, $b(f\backslash v) = b(f)/v$ and $b(f/v) = b(f)\backslash v$.*

3.2 Polyhedra of gradual functions

We assign the following $(n+1)$-dimensional polyhedra to a gradual set function f:

$$\begin{aligned}
P(f) &:= \{(y,\beta) \in \mathbb{R}^{n+1} :\ 0 \le y \le 1,\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\},\\
Q(f) &:= \{(y,\beta) \in \mathbb{R}^{n+1} :\ y \ge 0,\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\},\\
R(f) &:= \{(y,\beta) \in \mathbb{R}^{n+1} :\ y(X) - \beta \ge f(X) \text{ for every } X \subseteq V\}.
\end{aligned}$$

Proposition 3.3. *If f is a gradual set function, then $Q(f) = P(f) + \mathbb{R}^n_+$.*

Proposition 3.4. *For a gradual function f, $\operatorname{vert}(P(f)) \supseteq \operatorname{vert}(Q(f)) \supseteq \operatorname{vert}(R(f))$.*

Proposition 3.5. *For any gradual function f, the following hold:*

$$P(f\backslash v) = \{(y,\beta) \in \mathbb{R}^{n-1+1} : (y,1,\beta) \in P(f)\}, \textit{ and}$$

$$P(f/v) = \{(y,\beta) \in \mathbb{R}^{n-1+1} : (y,0,\beta) \in P(f)\},$$

that is, both $P(f\backslash v)$ and $P(f/v)$ are facets of $P(f)$.

3.3 Ideal gradual set functions

Definition 3.6. The gradual set function f is called *ideal* if the polyhedron $P(f)$ is integral.

For a gradual set function f, let us define the following finite set of vectors in $\mathbb{R}^{n+1}$: $S(f) := \{(\chi_X, f(X)) :\ X \subseteq V\}$. We denote the set $S(f) - \operatorname{cone}\{(\mathbf{0}, -1)\}$ by $S^{\downarrow}(f)$. We note that the idealness of f is equivalent to $P(f) = \operatorname{conv}\{S^{\downarrow}(b(f))\}$.

Theorem 3.7. *For a gradual set function, the following are equivalent:*
(i) *f is ideal, that is, $P(f) = \operatorname{conv}\{S^{\downarrow}(b(f))\}$*
(ii) *$b(f)$ is ideal, that is, $P(b(f)) = \operatorname{conv}\{S^{\downarrow}(f)\}$*
(iii) *$R(f)$ is an integer polyhedron,*
(iv) *$Q(f)$ is an integer polyhedron*

Proposition 3.8. *If f is ideal, then any minor of it is also ideal.*

Definition 3.9. A gradual set function is called *minimally nonideal* (*mni*) if it is not ideal but every minor of it is ideal.

3.4 Twisting

Definition 3.10. Let f be a gradual set function on ground set V, and let U be a subset of V. The *twisting of f at U* is the set function f^U on ground set V defined by

$$f^U(X) := f(X \Delta U) + |X \cap U|.$$

Proposition 3.11. *Every twisting of a gradual set function is a gradual set function.*

Proposition 3.12. *For a set $U \subseteq V$ and an element $v \in V$ the following hold.*

$$(f\backslash v)^{U-v} \cong \begin{cases} f^U/v & \text{if } v \in U, \\ f^U\backslash v & \text{if } v \notin U, \end{cases} \qquad (f/v)^{U-v} \cong \begin{cases} f^U\backslash v & \text{if } v \in U, \\ f^U/v & \text{if } v \notin U. \end{cases}$$

Proposition 3.13. *Every twisting of an ideal or mni set function is also ideal or mni, respectively.*

3.5 Examples

Clutters

Let $\mathcal{C}$ be a clutter on ground set V. Let $\mathcal{C}^\uparrow$ denote the uphull of $\mathcal{C}$, that is, $\{X \subseteq V : \exists C \in \mathcal{C} : C \subseteq X\}$. We associate the gradual set function $f_\mathcal{C}$ to $\mathcal{C}$ which is 1 on the sets in $\mathcal{C}^\uparrow$ and 0 otherwise. It is easy to check that this works well with the minor operations: for any $v \in V$, $f_{\mathcal{C}\backslash v} = f_\mathcal{C}\backslash v$ and $f_{\mathcal{C}/v} = f_\mathcal{C}/v$. Likewise, one can check that the blocker $b(f_\mathcal{C})$ is a translation of the set function corresponding to the blocker of $\mathcal{C}$, namely $f_{b(\mathcal{C})}$.

Proposition 3.14. *A clutter $\mathcal{C}$ is ideal / mni if and only if $f_\mathcal{C}$ is ideal / mni.*

Matroid rank functions

Proposition 3.15. *Both the rank function and the corank function of a matroid are ideal.*

Nearly bipartite graphs

For a graph $G = (V, E)$ let f_G be the gradual set function on V which is 0 on the emptyset, 1 on the stable sets of G and 2 otherwise. The graph G is called *nearly bipartite* if for every node v, the graph $G[V - N(v)]$ is bipartite, where $N(v)$ is the closed neighbourhood of v.

Proposition 3.16. *Let f be a gradual function with values in $\{0,1,2\}$ such that $f(\emptyset) = 0$ and $f(v) = 1$ ($\forall v \in V$). Then f is ideal if and only if $f = f_G$ for a nearly bipartite graph G.*

A class of mni gradual set functions

Proposition 3.17. *The following gradual functions are mni, if $n \geq 3$:*

$$\theta_n(X) := \begin{cases} 0 & \text{if } X = \emptyset, \\ 2 & \text{if } X = V, \\ 1 & \text{otherwise.} \end{cases} \qquad \bar{\theta}_n(X) := \begin{cases} 0 & \text{if } X = \emptyset, \\ n-2 & \text{if } X = V, \\ |X|-1 & \text{otherwise.} \end{cases}$$

An mni set function with non-simple fractional vertex

If an mni set function $f_\mathcal{C}$ is defined by an mni clutter $\mathcal{C}$, then $Q(f_\mathcal{C})$ has a unique fractional vertex and it is simple. This does not hold however for arbitrary mni set functions; in the thesis we give an example f on a 5-element ground set for which the unique fractional vertex of $Q(f)$ is not simple.

3.6 Convex and concave gradual extensions

Let g be a function on a box B in $\mathbb{R}^n$. We call g *gradual* if for every $x, z \in B$ for which $x \le z$, $g(x) \le g(z) \le g(x) + \|z - x\|_1$ holds. We consider gradual extensions of a gradual set function f to the unit cube $\{x \in \mathbb{R}^n : \mathbf{0} \le x \le \mathbf{1}\}$.

Proposition 3.18. *The maximal convex extension of a gradual set function f to the unit cube $\{x \in \mathbb{R}^n : \mathbf{0} \le x \le \mathbf{1}\}$ is*

$$\hat{f}(z) := \min\{ \sum_{Y \subseteq V} \lambda_Y f(Y) : \lambda_Y \ge 0 \; \forall Y \subseteq V, \sum_{Y \subseteq V} \lambda_Y = 1, \sum_{Y \subseteq V} \lambda_Y \chi_Y = z \},$$

which is moreover gradual. The minimal convex gradual extension of f to the unit cube is

$$\tilde{f}(z) := \max\{ f(Y) + z(Y) - |Y| : Y \subseteq V \}.$$

Theorem 3.19. *For a gradual set function f, the following are equivalent.*

(i) f *is ideal,*

(ii) f *has a unique convex gradual extension to the unit cube,*

(iii) f *has a unique concave gradual extension to the unit cube,*

(iv) there exist set functions u and l for which

$$\begin{aligned} \mathrm{conv}(S(f)) = \{(x, \alpha) \in \mathbb{R}^{n+1} : \; & \mathbf{0} \le x \le \mathbf{1}, \\ & x(Y) - \alpha \le u(Y) \quad \forall Y \subseteq V, \\ & x(Y) - \alpha \ge l(Y) \quad \forall Y \subseteq V \}, \end{aligned}$$

(v)

$$\begin{aligned} \mathrm{conv}(S(f)) = \{(x, \alpha) \in \mathbb{R}^{n+1} : \; & \mathbf{0} \le x \le \mathbf{1}, \\ & x(Y) - \alpha \le |Y| - f(Y) \quad \forall Y \subseteq V, \\ & x(Y) - \alpha \ge b(f)(Y) \quad \forall Y \subseteq V \}. \end{aligned}$$

The thesis is based on the following publications

[1] András Frank, Tamás Király, Júlia Pap, and David Pritchard. Characterizing and recognizing generalized polymatroids. Preprint.

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